

Non-existence of universal R -matrix for some C^* -bialgebras

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Abstract

For a C^* -bialgebra A with a comultiplication Δ , a universal R -matrix of (A, Δ) is defined as a unitary element in the multiplier algebra $M(A \otimes A)$ of $A \otimes A$ which is an intertwiner between Δ and its opposite comultiplication Δ^{op} . We show that there exists no universal R -matrix for some C^* -bialgebras.

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1 Introduction

We have studied C^* -bialgebras and their representations. In this paper, we consider a universal R -matrix of a C^* -bialgebra. Since a C^* -bialgebra is not always a bialgebra in a sense of the purely algebraic theory of quantum groups, we generalize the definition of universal R -matrix to C^* -bialgebras. Next, we consider whether a certain C^* -bialgebra has a universal R -matrix or not. In this section, we show our motivation, definitions and main theorem.

1.1 Motivation

In this subsection, we roughly explain our motivation and the background of this study. Explicit mathematical definitions will be shown after § 1.2.

Let \mathcal{O}_* denote the direct sum of all Cuntz algebras except \mathcal{O}_∞ :

$$\mathcal{O}_* = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3 \oplus \mathcal{O}_4 \oplus \cdots \quad (1.1)$$

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where \mathcal{O}_1 denotes the 1-dimensional C^* -algebra \mathbf{C} for convenience. In [9], we constructed a non-cocommutative comultiplication Δ_φ of \mathcal{O}_* . The C^* -bialgebra $(\mathcal{O}_*, \Delta_\varphi)$ has no antipode (with a dense domain). With respect to Δ_φ , tensor product formulae of representations of \mathcal{O}_n 's are well studied [8, 11]. Details about $(\mathcal{O}_*, \Delta_\varphi)$ are will be explained in § 1.3.

On the other hand, in the theory of quantum groups, a universal R -matrix for a quasi-cocommutative bialgebra is important for an application to mathematical physics and low-dimensional topology [5, 6, 7]. Therefore, it is meaningful for a given bialgebra to find its universal R -matrix if it exists. However, C^* -bialgebra is not always a bialgebra in a sense of the theory of purely algebraic case [1, 7]. Therefore, notions in purely algebraic case without change can not be always applied to C^* -bialgebra (see also Remark 1.2(ii)). Related studies were also considered by Van Daele and Van Keer for Hopf $*$ -algebras [15].

Our interests are to define a notion of universal R -matrix of a C^* -bialgebra and to clarify whether $(\mathcal{O}_*, \Delta_\varphi)$ in (1.1) has a universal R -matrix or not. In this paper, we show the negative result, that is, $(\mathcal{O}_*, \Delta_\varphi)$ has no universal R -matrix. For this purpose, we show a statement about the non-existence of universal R -matrix of a general C^* -bialgebra (Lemma 2.1).

1.2 Definitions

In this subsection, we recall definitions of C^* -bialgebra, and introduce a universal R -matrix of a C^* -bialgebra. For two C^* -algebras A and B , let $\text{Hom}(A, B)$ and $A \otimes B$ denote the set of all $*$ -homomorphisms from A to B and the minimal C^* -tensor product of A and B , respectively. Let $M(A)$ denote the multiplier algebra of A .

At first, we prepare terminologies about C^* -bialgebra according to [9, 13, 14]. A pair (A, Δ) is a C^* -bialgebra if A is a C^* -algebra and $\Delta \in \text{Hom}(A, M(A \otimes A))$ such that Δ is nondegenerate and the following holds:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \quad (1.2)$$

We call Δ the *comultiplication* of A . Remark that we do not assume $\Delta(A) \subset A \otimes A$. Furthermore, A has no unit in general for a C^* -bialgebra (A, Δ) . From these, a C^* -bialgebra is not always a bialgebra in a sense of the purely algebra theory [1, 7].

According to [5, 7, 15], we introduce a unitary universal R -matrix and the quasi-cocommutativity for a C^* -bialgebra as follows.

Definition 1.1 *Let (A, Δ) be a C^* -bialgebra.*

(i) The map $\tilde{\tau}_{A,A}$ from $M(A \otimes A)$ to $M(A \otimes A)$ defined as

$$\tilde{\tau}_{A,A}(X)(x \otimes y) \equiv \tau_{A,A}(X(y \otimes x)) \quad (X \in M(A \otimes A), x \otimes y \in A \otimes A) \quad (1.3)$$

is called the extended flip where $\tau_{A,A}$ denotes the flip of $A \otimes A$.

(ii) The map Δ^{op} from A to $M(A \otimes A)$ defined as

$$\Delta^{op}(x) \equiv \tilde{\tau}_{A,A}(\Delta(x)) \quad (x \in A) \quad (1.4)$$

is called the opposite comultiplication of Δ .

(iii) A C^* -bialgebra (A, Δ) is cocommutative if $\Delta = \Delta^{op}$.

(iv) An element $R \in M(A \otimes A)$ is called a (unitary) universal R -matrix of (A, Δ) if R is a unitary and

$$R\Delta(x)R^* = \Delta^{op}(x) \quad (x \in A). \quad (1.5)$$

In this case, we state that (A, Δ) is quasi-cocommutative.

Remark 1.2 (i) The additional assumption of unitarity of a universal R -matrix is natural for $*$ -algebras.

(ii) If A is unital, then $M(A \otimes A) = A \otimes A$ and $\tilde{\tau}_{A,A} = \tau_{A,A}$. In addition, if (A, Δ) is quasi-cocommutative with a universal R -matrix R , then $R \in A \otimes A$. In the purely algebraic theory of quantum groups, a bialgebra has a unit by definition [7]. Hence the quasi-cocommutativity makes sense by using a universal R -matrix as an invertible element in the tensor square of the bialgebra. On the other hand, there is no unit in C^* -bialgebra in general by definition. If there is no unit, then there is no invertible element in the algebra. Hence the quasi-cocommutativity and universal R -matrix make no sense for C^* -bialgebras if one uses the purely algebraic axiom without change.

(iii) If (A, Δ) is cocommutative, then the unit of $M(A \otimes A)$ is a universal R -matrix of (A, Δ) . Hence (A, Δ) is quasi-cocommutative. On the other hand, if a quasi-cocommutative C^* -bialgebra (A, Δ, R) is not cocommutative, then R is not a scalar multiple of the unit of $M(A \otimes A)$. Therefore the non-quasi-commutativity is stronger than the non-cocommutativity as same as the purely algebraic case.

(iv) About examples of universal R -matrix in purely algebraic theory, see examples in Chap. VIII.2 of [7].

1.3 Main theorem

In this subsection, we show our main theorem. Before that, we explain the C^* -bialgebra $(\mathcal{O}_*, \Delta_\varphi)$ in [9] more closely. Let \mathcal{O}_n denote the Cuntz algebra for $2 \leq n < \infty$ [3], that is, the C^* -algebra which is universally generated by generators s_1, \dots, s_n satisfying $s_i^* s_j = \delta_{ij} I$ for $i, j = 1, \dots, n$ and $\sum_{i=1}^n s_i s_i^* = I$ where I denotes the unit of \mathcal{O}_n . The Cuntz algebra \mathcal{O}_n is simple, that is, there is no nontrivial two-sided closed ideal. This implies that any unital representation of \mathcal{O}_n is faithful.

Redefine the C^* -algebra \mathcal{O}_* as the direct sum of the set $\{\mathcal{O}_n : n \in \mathbf{N}\}$ of Cuntz algebras:

$$\mathcal{O}_* \equiv \bigoplus_{n \in \mathbf{N}} \mathcal{O}_n = \{(x_n) : \|(x_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty\} \quad (1.6)$$

where $\mathbf{N} = \{1, 2, 3, \dots\}$ and \mathcal{O}_1 denotes the 1-dimensional C^* -algebra for convenience. For $n \in \mathbf{N}$, let $s_1^{(n)}, \dots, s_n^{(n)}$ denote canonical generators of \mathcal{O}_n where $s_1^{(1)}$ denotes the unit of \mathcal{O}_* . For $n, m \in \mathbf{N}$, define the embedding $\varphi_{n,m}$ of \mathcal{O}_{nm} into $\mathcal{O}_n \otimes \mathcal{O}_m$ by

$$\varphi_{n,m}(s_{m(i-1)+j}^{(nm)}) \equiv s_i^{(n)} \otimes s_j^{(m)} \quad (i = 1, \dots, n, j = 1, \dots, m). \quad (1.7)$$

For the set $\varphi \equiv \{\varphi_{n,m} : n, m \in \mathbf{N}\}$ in (1.7), define the $*$ -homomorphism Δ_φ from \mathcal{O}_* to $\mathcal{O}_* \otimes \mathcal{O}_*$ by

$$\Delta_\varphi \equiv \bigoplus \{\Delta_\varphi^{(n)} : n \in \mathbf{N}\}, \quad (1.8)$$

$$\Delta_\varphi^{(n)}(x) \equiv \sum_{(m,l) \in \mathbf{N}^2, ml=n} \varphi_{m,l}(x) \quad (x \in \mathcal{O}_n, n \in \mathbf{N}). \quad (1.9)$$

Then the following holds: The pair $(\mathcal{O}_*, \Delta_\varphi)$ is a non-cocommutative C^* -bialgebra ([9], Theorem 1.1); There is no antipode for any dense subbialgebra of $(\mathcal{O}_*, \Delta_\varphi)$ ([9], Theorem 1.2(v)). About much further properties of $(\mathcal{O}_*, \Delta_\varphi)$, see [9, 12]. About a generalization of $(\mathcal{O}_*, \Delta_\varphi)$, see [10].

Our main theorem is stated as follows.

Theorem 1.3 *There is no universal R -matrix of $(\mathcal{O}_*, \Delta_\varphi)$.*

In [5], Drinfel'd constructed a universal R -matrix by taking a completion of a given bialgebra with respect to a certain topology (see also Chap. XVI of [7]). As an analogy of this, we propose the following problem for non-quasi-cocommutative C^* -bialgebras.

Problem 1.4 For a non-quasi-cocommutative C^* -bialgebra (A, Δ) (for example, $(\mathcal{O}_*, \Delta_\varphi)$), construct an extension $(\tilde{A}, \tilde{\Delta})$ of (A, Δ) such that $(\tilde{A}, \tilde{\Delta})$ is a quasi-cocommutative.

As a related topic, Drinfel'd's quantum double [5] is known as a method of construction of a braided Hopf algebra from any finite dimensional Hopf algebra with invertible antipode. Difficulties of application of this method to infinite dimensional case are discussed in [15].

In § 2, we will prove Theorem 1.3. In § 3, we will show another example of non-quasi-cocommutative C^* -bialgebra.

2 Proof of Theorem 1.3

We prove Theorem 1.3 in this section. For this purpose, we recall multiplier algebra and nondegenerate homomorphism, and show a lemma.

2.1 Multiplier algebra and nondegenerate homomorphism

For a C^* -algebra A , let A'' denote the enveloping von Neumann algebra of A . The *multiplier algebra* $M(A)$ of A is defined by

$$M(A) \equiv \{a \in A'' : aA \subset A, Aa \subset A\}. \quad (2.1)$$

Then $M(A)$ is a unital C^* -subalgebra of A'' . Especially, $A = M(A)$ if and only if A is unital. The algebra $M(A)$ is the completion of A with respect to the strict topology.

A $*$ -homomorphism from A to B is not always extended to the map from $M(A)$ to $M(B)$. If (\mathcal{H}, π) is a nondegenerate representation of A , that is, $\pi(A)\mathcal{H}$ is dense in \mathcal{H} , then there exists a unique extension of π in $\text{Hom}(M(A), \mathcal{B}(\mathcal{H}))$. We state that $f \in \text{Hom}(A, M(B))$ is *nondegenerate* if $f(A)B$ is dense in a C^* -algebra B . If both A and B are unital and f is unital, then f is nondegenerate. For a $*$ -homomorphism f from A to $M(B)$, if f is nondegenerate, then f is called a *morphism* from A to B [16]. If $f \in \text{Hom}(A, B)$ is nondegenerate, then we can regard f as a morphism from A to B by using the canonical embedding of B into $M(B)$. Each morphism f from A to B can be extended uniquely to a homomorphism \tilde{f} from $M(A)$ to $M(B)$ such that $\tilde{f}(m)f(b)a = f(mb)a$ for $m \in M(B)$, $b \in B$, and $a \in A$. If f is injective, then so is \tilde{f} .

2.2 A lemma

For a unitary element U in a unital $*$ -algebra \mathfrak{A} , we define the (inner) $*$ -automorphism $\text{Ad}U$ of \mathfrak{A} by $\text{Ad}U(x) \equiv UxU^*$ for $x \in \mathfrak{A}$.

Lemma 2.1 *Let (A, Δ) be a C^* -bialgebra. If (A, Δ) is quasi-cocommutative, then for any two nondegenerate representations π_1 and π_2 of A , $(\pi_1 \otimes \pi_2) \circ \Delta$ and $(\pi_2 \otimes \pi_1) \circ \Delta$ are unitarily equivalent where we write the extension of $\pi_i \otimes \pi_j$ on $M(A \otimes A)$ as $\pi_i \otimes \pi_j$ for $i, j = 1, 2$.*

Proof. Since $\pi_{ij} \equiv \pi_i \otimes \pi_j$ is also nondegenerate, we can extend π_{ij} on $M(A \otimes A)$ and use the same symbol π_{ij} for its extension for $i, j = 1, 2$. From this, $\pi_{ij} \circ \Delta$ is well-defined. Let \mathcal{H}_i denote the representation space of π_i for $i = 1, 2$ and let T denote the flip between $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}_2 \otimes \mathcal{H}_1$, which is a unitary operator. Then the following holds:

$$\pi_{21} \circ \Delta = \text{Ad}T \circ \pi_{12} \circ \Delta^{op}. \quad (2.2)$$

By assumption, there exists a universal R -matrix R of (A, Δ) . Define $S \equiv \pi_{12}(R)$. From (2.2),

$$\pi_{21} \circ \Delta = \text{Ad}T \circ \pi_{12} \circ (\text{Ad}R \circ \Delta) = \text{Ad}W \circ \pi_{12} \circ \Delta \quad (2.3)$$

where $W \equiv TS$. This means that $(\pi_2 \otimes \pi_1) \circ \Delta$ and $(\pi_1 \otimes \pi_2) \circ \Delta$ are unitarily equivalent. \blacksquare

From Lemma 2.1, we see that the study of tensor products of representations of a bialgebra (A, Δ) is useful to verify whether (A, Δ) is quasi-cocommutative or not. The idea of Lemma 2.1 is a modification of a well-known fact in the purely algebraic case. We explain it as follows: Let A be a bialgebra with a comultiplication Δ in a sense of purely algebraic theory [7]. If (A, Δ) has a universal R -matrix (where the assumption of “braided” is not necessary), then for any two representations π_1, π_2 of A , $(\pi_1 \otimes \pi_2) \circ \Delta$ and $(\pi_2 \otimes \pi_1) \circ \Delta$ are equivalent ([7], Proposition VIII.3.1(a)). In order to define $(\pi_1 \otimes \pi_2) \circ \Delta$ and $(\pi_2 \otimes \pi_1) \circ \Delta$ for the C^* -bialgebraic case, we assume that two representations are nondegenerate because neither $\pi_1 \otimes \pi_2$ nor $\pi_2 \otimes \pi_1$ can be always extended on $M(A \otimes A)$ without any assumption.

2.3 Proof of Theorem 1.3

Let P_n denote the canonical projection of \mathcal{O}_* to \mathcal{O}_n . We identify a representation π of \mathcal{O}_n with the representation $\pi \circ P_n$ of \mathcal{O}_* here. If π is a

nondegenerate representation of \mathcal{O}_n , then π is also a nondegenerate representation of \mathcal{O}_* . If two representations π_1 and π_2 of \mathcal{O}_n are not unitarily equivalent, then π_1 and π_2 are also not as representations of \mathcal{O}_* .

It is known that there exist two unital representations π_1 and π_2 of \mathcal{O}_2 (especially, π_1 and π_2 are nondegenerate) such that $(\pi_1 \otimes \pi_2) \circ \Delta_\varphi|_{\mathcal{O}_4}$ and $(\pi_2 \otimes \pi_1) \circ \Delta_\varphi|_{\mathcal{O}_4}$ are not unitarily equivalent from Example 4.1 of [8]. Hence $(\pi_1 \otimes \pi_2) \circ \Delta_\varphi$ and $(\pi_2 \otimes \pi_1) \circ \Delta_\varphi$ are not unitarily equivalent. From these and the contraposition of Lemma 2.1, the statement holds. \blacksquare

3 Example

We show another example of non-quasi-cocommutative C^* -bialgebra. We call a matrix A *nondegenerate* if any column and any row are not zero. For $1 \leq n < \infty$, let $\mathbf{M}_n(\{0, 1\})$ denote the set of all nondegenerate $n \times n$ matrices with entries 0 or 1. In particular, $\mathbf{M}_1(\{0, 1\}) = \{1\}$. Define

$$\mathbf{M}_*(\{0, 1\}) \equiv \cup \{\mathbf{M}_n(\{0, 1\}) : n \geq 1\}. \quad (3.1)$$

For $A = (a_{ij}) \in \mathbf{M}_n(\{0, 1\})$, let \mathcal{O}_A denote the Cuntz-Krieger algebra associated with A [4]. Define the direct sum \mathbf{CK}_* of $\{\mathcal{O}_A : A \in \mathbf{M}_*(\{0, 1\})\}$:

$$\mathbf{CK}_* \equiv \bigoplus_{A \in \mathbf{M}_*(\{0, 1\})} \mathcal{O}_A \quad (3.2)$$

where we define $\mathcal{O}_1 \equiv \mathbf{C}$ for convenience. Then \mathbf{CK}_* is a C^* -bialgebra ([10], Theorem 1.2) such that \mathcal{O}_* in (1.1) is a C^* -subbialgebra of \mathbf{CK}_* and it is also a direct sum component of \mathbf{CK}_* ([10], Theorem 1.3(iv)). From this, Lemma 2.1 and Theorem 1.3, the C^* -bialgebra \mathbf{CK}_* is not quasi-cocommutative.

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